COMPARISON OF GAUSSIAN ELIMINATION AND CHOLESKY DECOMPOSITION SOLVING A LINEAR SYSTEM OF EQUATION

BY

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DECLARATION

I Nzabonimpa Godfrey, declare that this is my original work and it has never been submitted in any University for any award.

gnature...... ~ .

Date: 09/03/22

3.

APPROVAL

This research report has been done under my supervision and has been submitted with my approval.

Akugizibwe Edwin (PhD)

(Supervisor)

Signature ....

Date: 10/03/22

DEDICATION

I dedicate this research report to my lovely parents Mr. _'!)Jrimun.ahoro Jonathan and Mrs. Mujawimana Jeninah, brothers Mr. Maniriho Robert, Kwizera George and Ryaruhanga Micheal, and sisters Nyiramutuzo Peninah, Kampire Jenifer and Nyiraboneza Peruth for the support and words of encouragement without forgetting my friends. May the almighty God reward you abundantly and I promise never to forget you for the unending love and care you have always shown to me.

Also I dedicate this report to my supervisor for his guidance, May the almighty Gd reward you abundantly.

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First and foremost, I thank God for the grace he has granted me throughout my studies. I pray to the almighty to bless abundantly all those that were on my side during the periphera_l moments of financial support during compilation of this report.

My gratefulness and sincerity goes to my entire family who has always been there for me since I was born, struggled to make my education career a success and for having the best parental care which has been vital throughout my education, I will always live to respect you.

I extend great thanks to my University Supervisor Dr. Akugizibwe Edwin for the work done in form of guidance and time dedicated towards the completion of this report

I would also like to thank fellow students for the time- and encouragement that has made this work a successful reality. More thanks go to all my lecturers who have helped me in one way or the other and who offered me an opportunity to explore my career.

God bless you abundantly.

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ABSTRACT

There are several methods of solving a system of linear equations. Some of which are direct methods and others are iterative methods. In this work, we study direct methods specifically Gaussian elimination and Cholesky decomposition and make a comparison between the two. It is found out that Gaussian elimination is an algorithm in linear algebra for solving a system of linear equations and can also be used to find the rank of a matrix, to calculate the determinant of a matrix, and to calculate the inverse of an invertible square matrix Whereas Cho!sky decomposition is a decomposition of a Hermitian, positive-definite matrix into the product of a lower triangular matrix and its conjugate transpose, which is useful for efficient numerical solutions.

CHAPTER ONE

INTRODUCTION

1.0 Introduction

This section presents the background of the study, statement of the problem. purpose of the study, objectives of the study and significance of the study.

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1.1 Background of the study

According to Noreen Jamil (2012), a system of equation is a set or collections of equations solved together. Collection of linear equations is- tenned as· system of linear equations. They are often based on same set of variables. Various methods have been evolved to solve a linear equations but there is no best method yet proposed for solving system of linear equations.

Among other method of solving system of equations, the Gaussian method and the Cholesky decomposition methods will be applied discussed in details.

According to Matinfar et al., (2008), linear system of equations is important for studying and solving a large proportion of the problems in many topics in applied mathematics and engineering. According to Atkinson (1985), systems of Simultaneous linear equations occur in solving problems in a wide variety of areas with respect to mathematics, statistics, physical quantities (examples are temperature, voltage, population management and displacement), Social sciences, engineering and business. They arise directly in solving real life problems.

The world sometimes reveals itself to us as observable relationships among the relevant variables. What it does. makes evident relationships that describe how both the variable and their rate of change affect each other.

Tri-diagonal linear systems of the equations can be solved on conventional serial machines in a time proportional N, where N is the number of equations. The conventional algorithms do not led themselves directly to parallel computation on computers of *the* ILLIACIV class, in the *sense* that they appear to be inherently serial. An efficient paraUel algorithm is presented in which computation time grows as log N. 'The algorithm is based on recursive doubling solutions of linear recurrence relations, and can be used to solve recurrence relations of all orders (Stone, 1973).

There are various methods in solving linear system of simultaneous equations. In numerical analysis the techniques and methods for solving system of linear equations belongs to two categories: direct and iterative methods. The direct methods obtain the exact solution (in real arithmetic) in finitely many operations whereas iterative method generate a sequence of approximations that only converge in the limit to the solution. The direct method falls· into two categories that is the Gaussian elimination method and Cholesky decomposition method. Some others are matrix inverse method and LU factorization method and Cramer's rule method and Crout elimination method (Dehghan et al., 2006). As the standard method for solving systems of linear equations, Gaussian elimination (GE) is one of the most important and ubiquitous numerical algorithms. However, its successful use relies on understanding its numerical stability properties and how to organize its computations for efficient

execution on modern computers. We give an overview of GE, ranging from theory to computation. We explain why GE computes an LU factorization and the various benefits of this matrix factorization viewpoint. Pivoting strategies for ensuring numerical stability are described. Special properties of GE for certain classes of structured· matrices are summarized (Higham, 2011).

In our day to day lives in various fields, systems of simultaneous linear equations are being applied in solving problems in our societies with respect to businesses. Many people use systems of simultaneous linear equations for budgeting. In fact. big companies estimate their budgets and the cost of their products using systems of simultaneous equations. This helps these companies to provide better rates to their customers, thus enabling them to perform well in the market. Not only in companies, we can also use it in schools, and ceremony parties in our homes. These systems help us in making various predictions on an everyday basis. When we are starting some projects in our homes, we can use these systems to predict how we will perform in the future, and we can also use them to predict the cumulative profits for each month although many real-world factors are considered while making the predictions, systems of simultaneous prove to be very helpful in such scenarios.

1.2 Statement of the problem

According to Maron (1982), to make the relationship that exist between variables explicit, we frequently attempt to make a mathematical model that will accurately reflect real life situation. Many mathematical models accurately reflect real life situation but many of them have the same basic structure although disparity in Symbolic rotation may be utilized, which can arise from economics and transportation. There are several methods of solving a system of linear equations some of which are direct while others are iterative methods. Thus a mathematical model of the direct method of using the Gaussian elimination and Cholesk:y decomposition are highly needed so as to elicit a relationship within a linear system of equations.

Most of many mathematical models arises from net flow from one point to another or in relationship to population growth, that is, number of individuals in a particular age group at a particular time; thus a need to use Direct methods of Gaussian Elimination and Cholesky decomposition are highly required during the solve for system of linear equations.

1.3 Objectives

1.3.1 General Objective

The main objective of the study is to compare the Gaussian elimination and Cholesky decomposition methods in solving linear systems of equations.

1.3.2 Specific Objectives

To find out;

i. how Systems of linear equations are grouped.

ii. the various methods of solving linear systems of equations.

iii. the relationship between Gaussian elimination and Cholesky decomposition methods in solving linear systems of equations.

1.4 Scope

The study is done by following the mathematical and computational procedures.

The study is exclusively on the direct methods of solving linear systems of equations using Gaussian elimination and Cholesky decomposition.

1.6 Significance

i. Provision of more literature or knowledge for future use systems of linear equations.
ii. Explanation of mathematical models which reflect real life problems especially in different ficlds of science.

CHAPTER TWO

LIERATURER REVIEW

2.1 Introduction

According to Noreen Jamil (2012), a system linear equation is a set or collections of equations solved together. Collection of linear equations is termed as system of linear equations. They are often based on same set of variables, for example 3x+2y-z=I. Various methods have been evolved to solve a linear equations but there is no best method yet proposed for solving system of linear equations.

Among other methods of solving system of equations, the Gaussian method and the Cholesky composition methods are applied and discussed in details.

2.2 Other methods of solving linear systems of equations.

2.2.1 Crout elimi'natfon

According to Gustavson et al., (1970), an efficient implementation of the Crout elimination method in solving large sparse systems of linear equations of arbitrary structure is described. A computer program, GNSO, by symbolic processing, generates another program, SOL VE which represents the optimal reduced Crout algorithm in the sense that only nonzero elements are stored and operated on. The method represented is particularly powerful when a system of fixed sparseness structure must be solved repeatedly with different numerical values. In practical examples, the execution of SOLVE was observed to be typically N times as fast as that of the full Crout fllgorithm, where N is the order of the system.

We propose a highly efficient algorithm for direct solution of a sparse N X N system of linear algebraic equations,

$$Ax=b, \qquad (1.1)$$

With an arbitrary zero/nonzero structure of A. this algorithm systematically exploits sparseness by eliminating unnecessary arithmetic operations and by storing information in a compact, directly accessible manner. It is based on the Crout method, which consists of (1) factoring A into a products.

$$A=LU, \qquad (l.2)$$

of a lower triangular matrix *L* and a unit upper matrix *U;* and (2) solving by back substitution the systems-

$Ly=b$ $Ux=y$ For y and x, (1.3) .

respectively. Consider, (1.4)

a1 IX! + G12X2 + a13X3 = *b* I

a21x1 + a2i.x2 + *a23/3* = *2*

Q3 IXI + GJ2X2 + G33X3 = b3

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

l~1 ~ ~1] , is the lower triangular *L* and
[

ls1 ls2

412
Uzz
0 Uz3 , is the upper triangular
433

Problem 1

Use Crout' s method to solve the system of equations

$x+$ 2y+z = 4 -$x+$

3y+2z= 3 $2x+y+z$

= 6

Solution

$Ax=b$

$$\begin{bmatrix} 1 & 2 & 1 \\ -1 & 3 & 2 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 6 \end{bmatrix}$$

From $LU = A$

[1~1 0 412 U13] [1 2

1 U22 423 = -1 3 4

l31 SZ 0 433. 2 1

[U11 U12 U13] [1 2

la l21u12 + Uzz lziU13 + Uz3 = -1 3 il

la1u l31u + lazu22 la1ua + laua zuaa. 2 1

Comparing corresponding entries left hand side with right hand side, we get;

$u_{11} = 1$, $u_{12} = 2$, $u = 1$, $l_{21} = -1$

$l_{21}u_{12} + u_{22} = 3$ .... (i)

substituting the values of $l_{21} = -1$ *and* $u_1 = 2$ *in equation(i), becomes;* $u_{22} = 5$

$l_{21}u_{13} + u_{23} = 2$ ....(ii)

also substituting the values of $u_{13} = 1$ *and* $l_{21} = -1$ *in equation(ii), becomes;* $u_{23} = 3$

l3mu=2..................... (iii)

substituting the value of $u_{11} = 1$ *to get the value of* l_{31}, *in equation(iii)becomes;*

$l_{31} = 2$

$l_{31}u_{12} + l_{32}u_{22} = 1$………… (iv)

substituting the values of $l_{31} = 2$, $u_2 = 2$, *and* $u_{22} = 5$ *in equation(iv)becomes;*

l,,=?l,=-0.6

$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 1$…….. (v)

substituting the values of $Z_{31} = 2$, $u_{13} = 1$, $1_{32} = -0.6$, *and* $u_{23} =$

3 *in equation(v)becomes;*

$13a = 0.8$

Therefore;

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -0.6 & 1 \end{bmatrix}, \text{ and } U = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 3 \\ 0 & 0 & 0.8 \end{bmatrix}$$

Now, $Ly = b$

Lety= m

!1 ~ 0] [a] = [4]
0 b 3
1] c] 6]

a [4]

2 -0.6

-a+b 3
2a- 0.6b+ c 6

comparing their entries;

$a = 4$ .... (1)

a= 4

_ ----- ----------------

$-a+b=3$ .... (2)

substituting the value of a $=4$, *equation2)becomes;* $b=7$

$2a-0.6b+c=6$ ....(3)

also substituting the values of a, and b, equation(3)becomes;

$c=2.2$

Therefore

:EE]

2.2

Now,Ux = y,where x =

1 2 1 [X] [4

[

0 5 3 y = 7

0 0 0.8 z 2.2

x + 2y + z] 14]

5y+3z = 7

0.8z 2.2

Comparing entries, we have;

$x+2y+z=4$ **..............** (a)

$5y+3z=7$ ....(b)

$0.8z=2.2$ .... (c)

From equation(c);

$z=2.75$

To get the value of y, we are going to substitute the value of z in equation (b), becomes; $y=-0.25$

Also to get the value ofx, we are going to substitute the values of y and z in equation (a), becomes;

$x = 1.75$

x] 11.75

Therefore; $y = -0.25$

Thus ; z 2.75

$x = \mathbf{1.75}$,

$y = -0.25$, and z

$= 2.75$

However there are some other methods which I studied in O'level senior two that can-also help us to solve the above problem. These methods are;

Elimination method. This is achieved by adding or subtracting equations from each· other in order to cancel out one of the variables.

Substitution method. This is achieved by isolating the other variable in an equation and then substituting values for these variables in other another equation. and

Comparing method.

For these methods one can help each other in solving a problem, but also every method can solve the problem itself without any assistance from others. In other words, a method is applicable in another method when solving a problem. Let us use one of these methods in solving problem 1 being assisted by another method.

Let's use Elimination method,

$$x + 2y + z = 4 \qquad (1)$$

$$-x + 3y + 2z = 3 \qquad (2)$$

$$2x + y + z = 6 \qquad (3)$$

From equations (1) and (2), we going to eliminate x by adding them, becomes;

$Sy + 3z = 7$ (4)

Also we are going get equations (1) and (3) eliminate x by subtracting 2[(d)] trom (3), becomes;

$3y + z = 2$ Again subtracting 2[(2)] from (3), (5)

becomes; $7y + Sz = 12$

(6)

Applying in Substitution method,

From equation (5), $z = 2 - 3y$ (7)

Substituting $z = 2 - 3y$ in any of equations (4), and (6). Let's use equation (4), we have;

$$5y + 3(2 - 3y) = 7$$

$$5y + 6 - 9y = 7 \quad -4y = 1$$

$$y = -\tfrac{1}{4} = -0.25$$

Also we substitute $y = -0.25$ in equation (7) in order to get the value of z, becomes;

$$z = 2 - 3(-0.25) \quad z = 2.75$$

Finally we shall substitute the values ofy and z in any of equations (1), (2), and 3) Let's use equation (1), becomes;

$$x + 2(-0.25) + 2.75 = 4 \quad x = 4 - 2.25$$

$$x = 1.75$$

.As we have seen Substitution method comes in, but not because Elimination has failed it would also finish to solve the problem itself. Substitution and elimination are simpler methods of solving equations.

7.

231TERA TIV:E METHODS:

According to Meurant (1999), iterative methods define a sequence of approximations that are expected to be closer and closer to the exact solution in some given norm, stopping the iterations using some predefined criterion, obtaining a vector which is only an approximation of the solution. We start from an approximation to the-true solution and if successful, obtain better approximations from a computational cycle repeated as often as may be necessary for achieving a required accuracy so that the amount of arithmetic depends upon the accuracy required. Iterative methods are used mainly in those problems for which convergence is known to be rapid and for systems of large order but with many zero coefficients.

Generally, these methods have more modest storage requirements than direct methods and may also be faster depending on the iterative method and the problem. They usually also have better vectorization and parallelization properties.

The Jacobi and Gauss-Seidel methods are examples of the iterative methods.

Suppose we want to solve a linear system

$$Ax = b, \qquad (3.1)$$

Where A is non-singular and b is given.

An Iterative method constructs a sequence of vectors $\{x\}$, $k = 0, 1, \ldots$ which is expected to converge towards x which is the solution of (3.1), x^0 being given. The method is said to be convergent if limk ..... 00II x - xkll = 0. Most classical iterative methods use a splitting of the matrix A, denoted as A= M-N,

Where M *is* a non-singular matrix. Then, the sequence X is defined by

$$MI=N+ b, \qquad (3.2)$$

And x^0 is given. It obvious that if this method is convergent, it converges towards the unique solution of (3.1). An interesting question is to determine conditions for this sequence of vectors to converge.

Let £k =x- be the error at iteration number k. As obviously $Mx= N +b$, we get

$$M(x - x^{k=1}) = N(x - x^k),$$

$\square^{k+1} = M^{-1} N \square^{k}.$

erating this result, we obtain the equation for the error

$$\square^{k+1} = (M^{-1}N)^{k+1}\square^{0}. \tag{3.3}$$

From equation (3.3), the iterative method converges for all starting vectors if and only if

$$\lim_{k\to\infty} (M^{-1}N)^{k} = 0$$

2.4 DIRECT METHODS.

According to Meurant (1999), direct methods obtain the solution after a finite number floating point operations by doing combinations and modifications of the given equations. Of course, as computer floating point operations can only be obtained to a certain given precision, the computed solution is generally different from the exact solution, even with a direct method.

2.4.1 Gaussian elimination

According to Judd et al, (1998), Gaussian elimination is a common direct method for solving a general linear system. We will first consider the solution for a simple class of problems, and then we will see how to reduce the general problem to the simple case.

The simple case is that of triangular matrices. A is lower triangular if all nonzero elements lie on or below the diagonal; that is A has the form

$$A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

Upper triangular matrices have all nonzero entries on or above the diagonal. A is a triangular matrix if it is either upper or lower triangular. A diagonal matrix has nonzero element only on the diagonal. Some important facts to remember are that a triangular matrix is nonsingular if only and only if all the diagonal elements are nonzero, and that lower (upper) triangular matrices are closed under multiplication and inversion.

~ systems in which A is triangular can be solved by back-substitution. Suppose that A is lower t:b-,C,,, :gular and nonsingular. Since all nondiagonal elements in the first row of A are zero, the first row's contribution to the system $Ax = b$ reduces to $ax = b$, which has the solution

$n = b/a$. With this solution for x in hand, we next solve for x. Row 2 implies the equation an aux= b, in which only x is not known. Proceeding down the matrix, we can solve for each component of x in sequence. More formally, back-substitution for a lower triangular matrix

is the following procedure:

$$x_1 = \frac{b_1}{a_{11}}, \tag{1}$$

$$z.t = a_{kk}^{-1}\left(b_k - \sum_{j=1}^{k-1} a_{kj} x_j\right), \qquad k = 2,3,\ldots,T. \tag{2}$$

which is always well-defined for nonsingular, lower triangular matrices. If A is upper triangular, we can similarly solve $Ax = b$ beginning with $x = b/a$.

To measure the speed of this solution procedure, we make an operation count

There are n divisions, $nn-1)/2$ multiplications, and $n(n-1)/2$ additions/subtracffons. Ignoring additions/subtractions and dropping terms of order less than n, we have an operation count of $n/2$. Therefore solving a triangular system can be done in quadratic time.

We will now use the special method for triangular matrices as a basis for solving general nonsingular matrices. To solve $Ax = b$ for general A, we first factor A into the product of two triangular matrices, $A = LU$ where L is lower triangular and U is upper triangular. This is called an LU decomposition of A. We then replace the problem $Ax= b$ with the equivalent problem

$L\,Ux = b$, which in turn reduces to two triangular systems, $Lz = b$ and $Ur= z$. Therefore, to find x, we first solve for z in $Lz = b$ and solve for x in $Ux = z$.

Gaussian elimination produces such an LU decomposition for any nonsingular A, proceeding row by row, transforming A into an upper triangular matrix by means of a sequence of lower triangular t:EJ:sformations. The first step focuses: on getting the first column into upper triangular form, replacing *a* with a zero for *i*= 2, ... ,*n*,.

Suppose that a11 *i*= 0 Ifwe define /[1]n = *ala,i* = 2, ...,*n*, and *cl-ii*= aij-/[1]nav, *j* = 1, ... ,*n*, then *ah* is zero for *i* = 2, ... ,*n*. Let *Am*= *A*. lfwe define a new matrix *A*<2) to have the same first row as *A*, **m,**= *a*\ for i,j= 2, ... ,n, and A(2\1 = 0, *i* = 2, ... ,n, then

I

r / 0 0 1

@12 ·.. in

2

-1} 0

@22 ,e

A(zl,

ha 0

"Note that we have premultiplied *A* by a lower triangular matrix to get *A*<2>. Proceeding column by column in similar fashion, we can construct a series of lower triangular matrices that replaces the elements below the diagonal with zeros. If d'kk -:/;. 0, we define

@k ii

j [.. = a} kk ' *j* = *k*,*i*= *k* + 1, ... , *n*, (3)

0, *otherwise*

k+i _ {at - *l~at*1, *i* = *k* + 1, ...,*n*,*j* = *k*, ... , *n*,

ail - (4)

k *aiJ*, *otherwise*

then we have defined a sequence of matrices such that

	0		0	0	0
4+1) *l*-	0	•	0	*a*+	0	*400.*
	0	...	*o*	*l!.k*	0

L(k)

The result is that *A*<*n*> is upper triangular, where the factorization

$$L^{(n-1)} L^{(n-2)} \dots L^2 L^1 A = A^{(n)} \equiv U$$

implies that $A = LU$ where $L =$ *(L"D...LOO)'* is also lower triangular.

Note that, there are two difficulties with Gaussian elimination.

First is the possibility that @a is zero, making equation (3) ill-defined. However, as long as A below aa will be nonzero and a rearrangement of rows will bring a nonzero element to the I"" diagonal position, allowing us to proceed; this is called pivoting. Even if da is not zero, a small value will magnify any numerical error from an earlier step. Therefore a good pivoting scheme *will* use a rearrangement that minimizes this potential problem. Since good pivoting schemes are complex, readers should not write their own *LU* codes but use the refined codes, such as the programs in LAPACK, which are available. (Anderson et al, 1992) describes this package. Second, round-off error can accumulate because of the mixture of additions and subtractions that occur in.

To measure the speed of this procedure, we again perform an operation count The factorization step involves roughly n^3 /3 multiplications and divisions. Solving the two triangular systems implicit in $LUx = b$ *uses* a total of *ft'* multiplications and divisions. The cost of solving a linear problem depends on the context. Since the factorization cost is borne once for any matrix A, if you want to solve the linear problem with m different choices of b, the total cost is *"ff'* /3 + mn^2• Therefore the fixed cost is cubic in the matrix dimension n, but the marginal cost is quadratic.

2.4.2 Cholesky decomposition method.

Cholesky factorization is often the most expensive step in numerically solving a positive definite linear system of equations, such as in solving least square problems in signal processing. Due to the inherently recursive computation process for Cholesky decomposition. it is very difficult to obtain acceleration by exploiting _parallelism on FPGAs. The associated division and square root operations in traditional standard Cholesky decomposition represent difficulties because of long latency and data dependency. By introducing an extra diagonal *matrix* into the standard Cholesky decomposition we propose that the Cholesky factorization can be realized by designing a single triangular linear equation solver implemented on FPGAs. This eliminates the division dependency and thus improves the data throughput as well as system performance. By exploiting parallelism and designing the dedicated process engine with deep pipelines on FPGAs, we can achieve a significant computation speedup (Yang et al., 2009).

According to Rakotonirina (2009), we prove that if the matrix of the linear system is symmetric, fe Cholesky decomposition can be obtained from the Gauss elimination method without pivoting, without proving that the matrix of the system is positive definite. This method is unique when **A** is positive definite; there is only one lower triangular matrix **L** with strictly positive diagonal entries such that **A=LL.** However, the decomposition need not be unique when A is· positive semidefinite. The converse holds trivially: if **A** can be written as **LL** for some invertible **L**, lower triangular or otherwise, then A is Hermitian and positive definite.

According to Dostal et al., (2011), the Cholesky decomposition of symmetric positive semidefinite *matrix A* is a useful tool for solving the related consistent system of linear equations or evaluating the action of a generalized inverse, especially when *A* is relatively large and sparse. To use the Cholesky decomposition effectively, it is necessary to identify reliably the positions of zero rows

or columns of the factors and to choose these positions so that the nonsingular submatrix of *A* of the maximal rank is reasonably conditioned.

According to Salmela et al., (2006, June), both the matrix inversion and solving a set of linear equations can be computed with the aid of the Cholesky decomposition. In this paper, the Cholesky decomposition is mapped to the typical resources of digital signal processors (DSP) and our implementation applies a novel way of computing the fixed-point inverse square root function. The presented principles result in savings in the number of clock cycles. As a result; the Cholesky decomposition can be incorporated in applications such as 3G channel estimator where short execution time is crucial.

According to Seeger (2004), if the system matrix is symmetric positive definite, it is almost always possible to use a representation based on the Cholesk:y decomposition which renders the same results (in exact arithmetic) at the same or less operational cost, but typically is much more numerically stable.

We discussed this method in details *in* chapter four.

Therefore; from this chapter I have seen that there are other methods that can help us to solve our problems in our day to day lives in various fields.

CHAPTER THREE

MATERIALS AND METHODS

3.1 Methods **that are used in solving Systems of Linear Equations**

1. Write a linear system of equation
2. Write down the algorithms of Gaussian elimination and Cholesky decomposition
3. Solve the system of equations using the above two alternative methods

8.

CHAPTER FOUR

RESULTS

U {:..,eduction

-...-,: ~"I' presents results of the study that is the comparison between the two methods direct c::er::,:,....:,:hat is Gaussian elimination and Cholesky methods.

~ ~n Elimination

--z ~-:od is used to transform the coefficient matrix in its upper triangular form While it is plugged in ::-:: ::....,s,,,nented matrix of the system of linear equations.

tsus consider the following system of linear n equation and n unknowns

a1134+ a+2a +⋯+ a+n34 = b, 021x4

+ a22 +⋯+ an34 = b

....'!.. ....

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i$$

... ...

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

Which can be written under matricial form Ax=b

where

$$A = (a_{ij})_{1\leq i,j\leq n}, X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

:f a_{11} * 0, Gaussian Elimination in the first column is written 4$, +

a! I , + ..+ 4! = »%

(1) a22 X2 + ... · + (0) a2n Xn = 2 b(1)

(1)x + + (1) _ b<1)

12 2 ... ain Xn - t · ·

(1) (1) {1}

42 + ...+ ah»n = bl

. h (Q) (l) aLJaLJ-a11atJ

Wit a_{11} = a_11 and a_{11} = - --- ..... which may be written metrically A,X = b_1 and can be au obtained in Multiplying (1) by the lower triangular matrix.

$$G_1=\begin{pmatrix}1&0&0&\cdots&0\\ -\frac{a_{21}^{(0)}}{a_{11}^{(0)}}&1&0&\cdots&0\\ -\frac{a_{31}^{(0)}}{a_{11}^{(0)}}&0&1&\cdots&0\\ \vdots&&&\ddots&0\\ -\frac{a_{n1}^{(0)}}{a_{11}^{(0)}}&0&0&\cdots&1\end{pmatrix}$$

$A_1 = G_1A$ and $b_1 = G_1b$

The Gauss Elimination at the second column can be obtained by multiplying to the relation to

the lower triangular matrix

$$G_2=\begin{pmatrix}1&0&0&0&\cdots&0\\ 0&1&0&0&&0\\ 0&-\frac{a_{32}^{(1)}}{a_{22}^{(1)}}&1&0&&0\\ 0&-\frac{a_{42}^{(1)}}{a_{22}^{(1)}}&0&1&&0\\ \vdots&\vdots&&&\ddots&\\ 0&-\frac{a_{n2}^{(1)}}{a_{22}^{(1)}}&0&0&&1\end{pmatrix}$$

The Matricial equation becomes

$A_2X = b_2$

with $A_2 = G_2G_1A$ and $b_2 = G_2G_1b$.

In continuing so, finally, $UX = b_1$

with $U = G_{n-1}\cdots G_2G_1A$ an upper triangular matrix and $b_1 = G_{n-1}\cdots G_2G_1b$ so,

$G_{n-1}G_{n-2}\cdots G_2G_1$ is an invertable matrix and the equaton becomes $LUX = b$

with $L = G_1^{-1}G_2^{-1}\cdots G_{n-1}^{-1}G_{n-2}^{-1}$ and we have the decomposition of A as product oflower triangular matrix by an upper triangular matrix

$A = LU$

One can check easily that

$$L=\begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{a_{21}^{(0)}}{a_{11}^{(0)}} & 1 & 0 & 0 \\ \frac{a_{31}^{(0)}}{a_{11}^{(0)}} & \frac{a_{32}^{(1)}}{a_{22}^{(1)}} & \ddots & 0 \\ \frac{a_{(l+1)1}^{(0)}}{a_{11}^{(0)}} & \frac{a_{(l+1)2}^{(1)}}{a_{22}^{(1)}} & 1 & 0 \\ \frac{a_{n1}^{(0)}}{a_{11}^{(0)}} & \frac{a_{n2}^{(1)}}{a_{22}^{(1)}} & \frac{a_{nl}^{(l-1)}}{a_{ll}^{(l-1)}} & 1 \end{pmatrix}$$

Problem **4.1.1**

Uncle Innocent has two children: Martin and Janet Martin is 4 years older than Janet, the sum of their ages is 22. How old is each child?

Solution.

Let ***M*** be Martin's age and] be Janet's age.

The system of linear equation that models the facts given about the children's ages;

$$M=J+4$$

$$M-J=4 \qquad (1)$$

$$M+J=22 \qquad (2)$$

The first step is to write the coefficients of the unknowns in a matrix:

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

This is called the **coefficient matrix** of the system.

Next, the coefficient matrix is augmented by writing the constants that appear on the right□hand sides of the equations as an additional column:

$$\begin{pmatrix} 1 & -1 & 3 \\ 1 & 1 & 22 \end{pmatrix}$$

Tr:::5 is called the **augmented matrix,** and each row corresponds to an equation in the given system.

~first row, **R**= (l,'--1, 4), corresponds to the first equation, $M - J = 4$ and

the second row, ***R,*** = (1, 1, 22), corresponds to the second equation, $M + J = 22$.

You may choose to include a vertical line as shown above to separate the coefficients of the unknowns from the extra column representing the constants.

Now, the counterpart of eliminating a variable from an equation in the system is changing one of the entries in the coefficient matrix to zero. Likewise, the counterpart of adding a multiple of one equation to another is adding a multiple of one row to another row.

$$\begin{pmatrix} 1 & -1 & 4 \\ 1 & 1 & 22 \end{pmatrix}$$

Adding -1 times the first row of augmented matrix to the second row yields;

$$-R_1 + R_2 \rightarrow R_2$$

$$\begin{pmatrix} 1 & -1 & 4 \\ 0 & 2 & 18 \end{pmatrix}$$

The new second row translates into $2J = 18$, which means $J = 9$. Back-substitution into the first row (that is, into the equation that represents the first row) yields $M - 1(9) = 4$ which means $M = 4 + 9 = 13$

Therefore; the solution to the system: $\begin{pmatrix} ? \\ 7 \end{pmatrix} = \begin{pmatrix} 13 \end{pmatrix}$

Thus Martin is 13 years old, and his sister Janet is 9 years old .

42 Cholesky **Decomposition**

Nini' suppose A is a symmetric positive definite matrix. Then, the Cholesky method consists to

decompose A as the product

.£. $= GTG$

with G is an upper triangular matrix and GT its transpose. Let

us at first generalize this decomposition.

Definition; Let $A=$ *(aiJ)isiJsn* a complex symmetric matrix. Let us call

Gauss matrix of A the upper triangular matrix $U(A)$, obtained after transforming A by

the Gauss elimination above.

$$U(A) = \begin{pmatrix} u_{11} & & u_{1n} \\ & & u_{2n} \\ & & \vdots \\ & 0 & u_{nn} \end{pmatrix}$$

The Gauss Matrix of A . Then A can be decomposed as the product $A = GT\,G$

$$G= \begin{pmatrix} \frac{u1}{7} & \frac{412}{7} & \frac{u_{1n}}{\sqrt{u_{11}}} \\ 0 & \frac{U22}{V022} & \frac{u_{2n}}{\sqrt{u_{22}}} \\ 0 & 0 & \frac{u_{nn}}{\sqrt{u_{nn}}} \end{pmatrix}$$

with ,$[ur$oot squared of the complex number u

2of,

$$U(A) = \begin{vmatrix} 0 & 0 & & 0 \\ i & @12 & & in \\ & 1 & & 1 \\ 0 & azz & & llzn \\ & & \ddots & \\ & & & (n-1) \\ 0 & 0 & \cdots & llitri \end{vmatrix}$$

$$(A)= \begin{vmatrix} 1 & 0 & 0 & 0 \\ g_{21} & & & \\ (0) & 1 & 0 & 0 \\ Cli1 & & & \\ (0) & (1) & & \\ da4 & (tjz & & \\ (0) & (1) & & \\ & @\pm2 & & \\ & & & 0 \\ (0) & (1) & & \\ a(l+1) & Gt+12 & 1 & 0 \\ & (1) & & \\ 44\% & Ha2 & & \\ (0) & (1) & (L-1) & \\ Gu & a(n)2 & GR & 1 \\ (0) & (1) & (L....:1) & \\ a & a22 & a & \end{vmatrix}$$

$A= L(A)D\ 'DU(A)$

With

$$D= \begin{vmatrix} 1 & & & 0 \\ & 2 & \ddots & \\ & & & \frac{1}{\sim} \\ 0 & 0 & & n \end{vmatrix}$$

Let $G = DU(A).$ Since A is symmetric, hence $L(A)D\ ' GT\cdot$

Problem 4.2.1

Solve the system of equations:

4x,+12x,- 16x4 = 2 12x1

+ 37x,- 43x4 = 8 -16x1

-43x2 + 98x = 11

Solution

98 11

Let A= (3

-16 -43

Since A is symmetric, then $A = LLT$

Steps to follow when solving this problem;

Step 1: $A = LL'$ then solve for L

Step2: $AX= B$

But $A= LLr.$ becomes $LLTX = B$ We

let $Lr X = Y$ then we solve for Y Step3:

$Lr X = Y$ we solve for X

$= LL'X =$ A,let $L'X = Y = LY = b$ then $I' X = b$

o

Let L = (: c c

e o

From $A= LLT$ we have 24

12 o b
12 37 43 = b c c
-16 -43 98 d e o

$$\begin{pmatrix} 4 & 12 & -16 \\ 12 & 37 & -43 \\ -16 & -43 & 98 \end{pmatrix} = \begin{pmatrix} a^2 & ab, & ad \\ ab & b^2+c^2 & bd+ce \\ ad & bd+ce & d?+e?+f? \end{pmatrix}$$

Equating co-efficiencies.

a°=4

$\Rightarrow a=2$

:zb=12

$\Rightarrow b=6$ cd

= -16

=d=-8

$bd + ce = -43$

~ d = 6. -8 + e = -43

$\Rightarrow$-48+e = -43

e=5

$b^2 + c^2 = 37$

36 +c2 = 37 >cd

= 1

c= 1 d?+e?+f?=

98

$\Rightarrow$ -82 = 52 + 12 = 98

> 89 + 1^2 = 98

> f?=9

$\Rightarrow$[=3

Therefore,

$a = 2,\ b = 6,\ c = 1,\ d = -8,\ e = 5,\ and\ f = 3$

So $L = \begin{pmatrix} 2 & 0 & 0 \\ 6 & 1 & 0 \\ -8 & 5 & 3 \end{pmatrix}$ and $L^T = \begin{pmatrix} 2 & 6 & -8 \\ 0 & 1 & 5 \\ 0 & 0 & 3 \end{pmatrix}$

From $LY = b$

(2 0 0) ~1) '(2)

6 1 O Y1 = 8

-8 5 3 1 11

By forward substitution we have,

2y= 2

y,= 1

6y+ Y1 = 8

>6+y,=8

>y,=2

-8y+5y, +3y= 11

$\Rightarrow y_3 \Rightarrow 3$10 + 3y3 = 11

Again from $L^r X = y$ we have,

(2 6 -8) (X1) (1)

0 1 5 x_2 = 2

0 0 3 x_3 3

By backward substitution, we have,

$3x_3 = 3$

$\Rightarrow x_3 = 1$

$x_2 + 5x_3 = 2$

$\Rightarrow x_1 + 5 = 2$

$\Rightarrow\ x_2 = -3$

$2x_1 + 6x_2 - 8x_3 = 3$

> 2x,+ 6.-3- 8.1 = 1

>2x,= 27

$\Rightarrow$ x1 = **13.5**

Therefore, *(a42,33)* = (13.5,-3,1)

This method cannot help us to solve the **Problem 4.1.1,** because the system matrix is not symmetric positive definite.

Therefore; from this chapter I have seen that there are some problems which Cholesky method cannot help us to solve.

CHAPTER FIVE

DISCUSSION, CONCLUSION AND RECOMMENDATION OF FINDINGS

5.0 Introduction

This chapter is about application of Gaussian elimination and Cholesky decomposition methods to linear system of equations

*5.1*Relationship between Gaussian elimination and Cholesky decomposition methods in solving linear systems of equations

5.1.1 Applications of Gaussian elimination 1.
Computing **determinants**

To explain how Gaussian elimination allows the computation of the determinant of a square matrix, we have to recall how the elementary row operations change the determinant: **9'**

Swapping two rows multiplies the determinant by -1

4 Multiplying a row by a nonzero scalar multiplies the determinant by the same scalar 4

Adding to one row a scalar multiple of another does not change the determinant.

If Gaussian elimination is applied to a square matrix *A* produces a row echelon matrix *B*, let *d* be the product of the scalars by which the determinant has been multiplied, using the above rules. Then the determinant of *A* is the quotient by *d* of the product of the elements of the diagonal of *B.* Computationally, for an *n x n* matrix, this method needs only *O(n?)* arithmetic operations, while using Leibniz formula for determinants requires O(n!) operations (number of summands in the formula), and recursive Laplace expansion requires *0(2n)* operations (number of sub-determinants to' compute, if none is computed twice). Even on the fastest computers, these two methods are impractical or almost impracticable for *n* above 20.

2. Finding the inverse of a matrix

A variant of Gaussian elimination called Gauss-Jordan elimination can be used for finding the inverse of a matrix, if it exists. If *A* is an *n x n* square matrix, then one can use row reduction to compute its inverse matrix, if it exists. First, the *n x n* identity matrix is augmented to the right of *A*, forming an *n x 2n* block matrix [A/I]. Now through application of elementary row operations, find the reduced echelon form of this *n x 2n* matrix. The matrix *A* is invertible if and only if the left block can be reduced to the identity matrix I; in this case the right block of the final matrix is A, IR the algorithm is unable to reduce the left block to I, then A is not invertible.

3. **Computing ranks and bases**

The Gaussian elimination algorithm can be applied to any m n matrix A. In this way, for example, some 6x9 matrices can be transformed to a matrix that has a row echelon form like where the stars are arbitrary entries, and a, b, c, d, e are nonzero entries. This echelon matrix T contains a wealth of information about A: the rank of A is 5, since there are 5 nonzero rows in T; the vector space spanned by the columns of A has a basis consisting of its columns 1, 3, 4, 7 and 9 (the columns with a, b, c, d, e in T), and the stars show how the other columns of A can be written as linear combinations of the basis columns. This is a consequence of the distributivity of the dot product in the expression of a linear map as a matrix. All of this applies also to the reduced row echelon form, which is a particular row echelon format.

4. **Computational efficiency**

According to Rani, N. S, the number of arithmetic operations required to perform row reduction is one way of measuring the algorithm's computational efficiency. For example, to solve a system of n equations for n unknowns by performing row operations on the matrix until it is in echelon form, and then solving for each unknown in reverse order, requires $n(n+1)/2$ divisions, $(2n^3 + 3n - 5n)/6$ multiplications, and $(2n^3 + 3n^2 - 5n)/6$ subtractions, for a total of approximately $2n^3/3$ operations. Thus it has arithmetic complexity of On'); This arithmetic complexity is a good measure of the time needed for the whole computation when the time for each arithmetic operation is approximately constant. This is the case when the coefficients are represented by floating-point numbers or when they belong to a finite field If the coefficients are integers or rational numbers exactly represented, the intermediate entries can grow exponentially large, so the bit complexity is exponential. However, there is a variant of Gaussian elimination, called the Bareiss algorithm, that avoids this exponential growth of the intennediate entries and, with the same arithmetic complexity of O(n'), has a bit complexity *of On?).* This algorithm can be used on a computer for systems with thousands of equations and unknowns. However, the cost becomes prohibitive for systems with millions of equations. These large *systems* are generally solved using iterative methods. Specific methods exist for systems whose coefficients follow a regular pattern.

9.

To put an nXn matrix into reduced echelon form by row operations, one needs n^3 arithmetic operations, which is approximately 50% more computation steps.

One possible problem is numerical instability, caused by the possibility of dividing by very small numbers. If, for example, the leading coefficient of one of the rows is very dose to zero, then to row-reduce the matrix, one would need to divide by that number. This means that any error existed for the number that was close to zero would be amplified. Gaussian elimination is numerically stable for diagonally dominant or positive-definite matrices. For general matrices, Gaussian elimination is usually considered to be stable, when using partial pivoting, even though there are examples of stable matrices for which it is unstable.

5. Generalizations

Gaussian elimination can be performed over any field, not just the real numbers.

Buchberger's algorithm is a generalization of Gaussian elimination to systems of polynomial equations. This generalization depends heavily on the notion of a monomial order. The choice of an ordering on the variables is already implicit in Gaussian elimination, manifesting as the choice to work from left to right when selecting pivot positions.

Computing the rank of a tensor of order greater than 2 is NP-hard. Therefore, if *Pt* NP, there cannot **be a** polynomial time analog of Gaussian elimination for higher-order tensors (matrices are array representations of order-2 tensors).

5.1.2 Applications of Cholesky decomposition

The Cholesky decomposition is mainly used for the numerical solution of linear equations . If **A** is symmetric and positive definite, then we can solve by first computing the Cholesky decomposition, then solving for y by forward substitution, and finally solving for **x** by back substitution. An alternative way to eliminate taking square roots in the decomposition is to compute the Cholesky decomposition, then solving for y, and finally solving. For linear systems that can be put into symmetric form, the Cholesky decomposition (or its LDL variant) is the method of choice, for superior efficiency and numerical stability. Compared to the LU decomposition, it is roughly twice as efficient

1. Linear least squares

Systems of' the form $AX = b$ with A symmetric and positive definite arise quite often in applications. For instance. the normal equations in linear least squares problems are of this form. It may also happen that matrix **A** comes from an energy functional, which must be positive from physical considerations; this happens frequently in the numerical solution of partial differential equations.

2. Monte Carlo simulation

The Cholesky decomposition is commonly used in the Monte Carlo method for simulating systems with multiple correlated variables. The covariance matrix is decomposed to give the lm.ver-triangular L. Applying this to a vector of uncorrelated samples U produces a sample vector LU with the covariance properties of the system being modelled.

The following simplified example shows the economy one gets from the Cholesky decomposition: suppose the goal is to generate two correlated normal variables and with given correlation coefficient. To accomplish that. it is necessary to first generate two uncorrelated Gaussian random variables and. which can be done using a Box--Muller transform. Given the required correlation coefficient, the correlated normal variables can be obtained via the transformations and.

3. Kalman filters

Unscented Kalman filters commonly use the Cholesk:y decomposition to choose a set of so-called sigma points. The Kalman filter tracks the: average state of a system as a vector x of length N and covariance as an NxN matrix **P.** The matrix **P** is always positive semi-definite and can be decomposed into **LL'.** The columns of L can be added and subtracted from the mean **x** to form a set of $2N$ vectors called sigma points. These sigma points completely capture the mean and covariance ofthe system state.

4. Matrix inversion

The explicit inverse of a Hermitian matrix can be computed by Cholesky decomposition, in a manner similar to solving linear systems, using operations (multiplications). The entire inversion can even be efficiently performed in-place. A non-Hermitian matrix **B** can also be inverted using the following identity, where **BB** will always be Hennitian.

5. Computation

There are various methods for calculating the Cholesky decomposition. The computational complexity of commonly used algorithms is *On)* in general. The algorithms described below all involve about *n/3* FLOPs ($n^3/6$ multiplications and the same number of additions), where *n* is the size of the matrix **A.** Hence, they have half the cost of the LU decomposition, which uses $2n^3/3$ FLOPs (Trefethen and Bau 1997).

5.3 Conclusion

Gaussian elimination, also known as **row reduction,** is an algorithm in linear algebra for solving a system of linear equations. It is usually understood as a sequence of operations performed on the corresponding matrix of coefficients. This method can also be used to find the rank of a matrix, to ca1cu1ate the determinant of a matrix. and to calculate the inverse of an invertible square matrix.

Whereas

Cholesky decomposition or **Cholesky factorization** is a decomposition of a Hermitian, positive-definite matrix into the product of a lower triangular matrix and *its* conjugate transpose, *which is* useful for efficient numerical solutions.

The results show that Cholesky decomposition involves a lot of computation as compared to Gaussian elimination in solving linear systems of equation. I therefore conclude that Gaussian elimination method should be used to solve linear systems of equation because it involves little computations.

5.4 Recommendation.

Gaussian elimination should be used over Cholesky decomposition because it involves few iterations compared to Cholesky decomposition.

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